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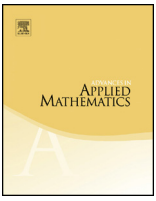

Bijective proofs of shuffle compatibility results

Duff Baker-Jarvis [a], Bruce E. Sagan [b,*]

[a] *Department of Mathematics and Statistics, Wake Forest University, Winston-Salem, NC 27109, USA*
[b] *Department of Mathematics, Michigan State University, East Lansing, MI 48824-1027, USA*

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ABSTRACT

Define a permutation to be any sequence of distinct positive integers. Given two permutations π and σ on disjoint underlying sets, we denote by $\pi \sqcup\!\sqcup \sigma$ the set of shuffles of π and σ (the set of all permutations obtained by interleaving the two permutations). A permutation statistic is a function St whose domain is the set of permutations such that $\mathrm{St}(\pi)$ only depends on the relative order of the elements of π. A permutation statistic is shuffle compatible if the distribution of St on $\pi \sqcup\!\sqcup \sigma$ depends only on $\mathrm{St}(\pi)$ and $\mathrm{St}(\sigma)$ and their lengths rather than on the individual permutations themselves. This notion is implicit in the work of Stanley in his theory of P-partitions. The definition was explicitly given by Gessel and Zhuang who proved that various permutation statistics were shuffle compatible using mainly algebraic means. This work was continued by Grinberg. The purpose of the present article is to use bijective techniques to give demonstrations of shuffle compatibility. In particular, we show how a large number of permutation statistics can be shown to be shuffle compatible using a few simple bijections. Our approach also leads to a method for constructing such bijective proofs rather than having to treat each one in an ad hoc manner. Finally, we are able to prove a conjecture of Gessel

* Corresponding author.
E-mail addresses: bjduff@gmail.com (D. Baker-Jarvis), sagan@math.msu.edu (B.E. Sagan).

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and Zhuang about the shuffle compatibility of a particular statistic.

1. Introduction

Let $\mathbb{P}$ and $\mathbb{N}$ be the positive and nonnegative integers, respectively. To denote the cardinality of a set U we use $\#U$ or $|U|$. All subsets of $\mathbb{P}$ should be assumed to be finite unless otherwise noted. For $U \subset \mathbb{P}$, a *permutation* of U is a linear order $\pi = \pi_1\pi_2\dots\pi_n$ of the elements of U. We sometimes separate the elements of π by commas for ease of reading. Denote the set of all linear orders on U by

$$L(U) = \{\pi \mid \pi \text{ is a linear order on } U\}.$$

The *length* of a permutation is the cardinality of its underlying set, i.e. $|U|$, which we denote by $|\pi|$. The *ground set* of a permutation $\pi \in L(U)$ is the set U, and we write $\mathrm{gs}(\pi) = U$.

For $n, i, j \in \mathbb{P}$ we use the notation $[n] = \{1, 2, \dots, n\}$, and $[i, j] = \{i, i+1, \dots, j\}$. Also let $[n] + i = \{k + i \mid k \in [n]\}$. For sets, $U \sqcup V = W$ indicates that W is the disjoint union of U and V. Double braces indicate a multiset, that is, a family of elements where repetition is allowed. We will sometimes use multiplicity notation for multisets, e.g., $\{\{1, 2^4, 3^3\}\}$ is the multiset that contains 1, 2 four times, and 3 three times.

To compare permutations on different sets of the same size we have the following definition.

Definition 1.1. Let $U, V \subseteq \mathbb{P}$ be two subsets of the positive integers such that $|U| = |V|$. Let $\pi \in L(U)$. Define the *standardization to* V of $\pi = \pi_1\pi_2\dots\pi_\ell$ to be

$$\mathrm{std}_V(\pi) = f(\pi_1)f(\pi_2)\dots f(\pi_\ell)$$

where $f : U \to V$ is the unique strictly increasing bijection. Let $n = |U|$. Then, if no subscript is given, define

$$\mathrm{std}\,\pi := \mathrm{std}_{[n]}(\pi)$$

to be the standardization of π to $\{1, 2, \dots, |U|\}$. □

Two permutations $\pi \in L(U)$, and $\pi' \in L(V)$ are said to have the same *relative order* if $\mathrm{std}_V(\pi) = \pi'$, or equivalently, $\mathrm{std}_U(\pi') = \pi$. For example if $U = \{1, 7, 8\}$, $V = \{2, 3, 9\}$, $\pi = 781$, $\pi' = 392$, then π and π' have the same relative order since $\mathrm{std}_U(392) = 781$. Equivalently, $\mathrm{std}\,\pi = \mathrm{std}\,\pi' = 231$.

A *permutation statistic* is a map St with domain

$$\bigsqcup_{\substack{U\subset\mathbb{P}\\|U|<\infty}} L[U]$$

such that whenever π and π' have the same relative order, then $\mathrm{St}(\pi) = \mathrm{St}(\pi')$. It is useful to extend the notation for permutation statistics to sets of permutations by defining, for a set of permutations Π, $\mathrm{St}(\Pi)$ to be the multiset

$$\mathrm{St}(\Pi) = \{\{\mathrm{St}(\pi) \mid \pi \in \Pi\}\}.$$

We call this multiset the *distribution* of St over the set Π.

The basic example of a (set-valued) permutation statistic is that of the descent set, Des. For a permutation π, a *descent* of π is a position i such that $\pi_i > \pi_{i+1}$. Then the *descent set of* π is

$$\mathrm{Des}(\pi) = \{i \mid i \text{ is a descent of } \pi\}.$$

The *descent number* of π is $\mathrm{des}(\pi) = \#\,\mathrm{Des}(\pi)$. Another important statistic is the *major index*, maj, given by

$$\mathrm{maj}(\pi) = \sum_{i\in\mathrm{Des}(\pi)} i.$$

For example, given the permutation $\pi = 2157364 \in L([7])$ we have $\mathrm{Des}(\pi) = \{1, 4, 6\}$ and $\mathrm{maj}(\pi) = 1 + 4 + 6 = 11$.

We call a permutation statistic, St, a *descent statistic* if it is a permutation statistic such that $\mathrm{Des}(\pi) = \mathrm{Des}(\pi')$ implies $\mathrm{St}(\pi) = \mathrm{St}(\pi')$. Both Des and maj are examples of descent statistics. There are many permutation statistics in the literature which are not descent statistics. One such statistic is

$$\mathrm{inv}(\pi) = \#\{i < j \mid \pi_i > \pi_j\}$$

which counts inversions in a permutation. For example 132 and 231 are two permutations that have the same descent set $\{2\}$, but $\mathrm{inv}(132) = 1$ whereas $\mathrm{inv}(231) = 2$.

Given a permutation π, a *subword* of π is a subsequence of not necessarily consecutive elements, whereas a *factor* is a subsequence whose elements are consecutive. For two permutations with disjoint ground sets, a *shuffle* of π and σ is a permutation $\tau \in L(\mathrm{gs}(\pi) \sqcup \mathrm{gs}(\sigma))$ such that both π and σ occur as subwords. The *shuffle set* of π and σ is

$$\pi \sqcup\!\sqcup \sigma = \{\tau \mid \tau \text{ is a shuffle of } \pi, \sigma\}$$

which always has cardinality $\binom{|\pi|+|\sigma|}{|\pi|}$. As an example, if $\pi = 132$ and $\sigma = 76$ then

$$\pi \sqcup\!\sqcup \sigma = \{13276,\ 13726,\ 13762,\ 17326,\ 17362,\ 17632,\ 71326,\ 71362,\ 71632, 76132\}$$

which has size $\binom{3+2}{3} = 10$. Whenever we write a shuffle set $\pi \sqcup\!\sqcup \sigma$ we will implicitly assume that the permutations π and σ have disjoint ground sets.

It is helpful to have a way to discuss and distinguish individual shuffles without carrying the entire information of both permutations along. To do this, we will use words in the two letter alphabet $A = \{a, b\}$. Denote by

$$A^* = \{\alpha_1\alpha_2 \dots \alpha_k \mid k \geq 0,\ \alpha_i \in A\}$$

the set of words in the letters of A. This is called the *Kleene closure* of A. Suppose that $|\pi| = m$ and $|\sigma| = n$. Then we will associate to each permutation $\tau \in \pi \sqcup\!\sqcup \sigma$ a word $\omega(\tau) \in A^*$ of length $m + n$ obtained by replacing the elements of π with the letter a and elements of σ with the letter b. Call $\omega(\tau)$ the *word of* τ. For example if $\pi = 132$, $\sigma = 4589$ and $\tau = 1453829$, then $\omega(\tau) = abbabab$. It is true that $\omega(\tau)$ depends on both π and σ, but these will always be clear from context. We now introduce the definition which will be our fundamental object of study.

Definition 1.2. Assume $\pi, \pi', \sigma, \sigma'$ are permutations such that $|\pi| = |\pi'|$, $|\sigma| = |\sigma'|$ and $\mathrm{gs}(\pi) \cap \mathrm{gs}(\sigma) = \mathrm{gs}(\pi') \cap \mathrm{gs}(\sigma') = \emptyset$. Call a permutation statistic St *shuffle compatible* if for all such permutations which also satisfy $\mathrm{St}(\pi) = \mathrm{St}(\pi')$ and $\mathrm{St}(\sigma) = \mathrm{St}(\sigma')$ we have

$$\mathrm{St}(\pi \sqcup\!\sqcup \sigma) = \mathrm{St}(\pi' \sqcup\!\sqcup \sigma').$$

Being shuffle compatible is also equivalent to the existence of a bijection $\Theta : \pi \sqcup\!\sqcup \sigma \to \pi' \sqcup\!\sqcup \sigma'$ with the property that $\mathrm{St}(\Theta(\tau)) = \mathrm{St}(\tau)$ for all $\tau \in \pi \sqcup\!\sqcup \sigma$. We call a map with this property St *preserving*. □

As an illustration of this, let us consider the maj statistic with $\pi = 4312$, $\pi' = 2341$, $\sigma = 76$ and $\sigma' = 98$. These satisfy $\mathrm{maj}(\pi) = \mathrm{maj}(\pi') = 3$ and $\mathrm{maj}(\sigma) = \mathrm{maj}(\sigma') = 1$. Then one can check that

$$\mathrm{maj}(\pi \sqcup\!\sqcup \sigma) = \mathrm{maj}(\pi' \sqcup\!\sqcup \sigma') = \{\{4, 5, 6^2, 7^2, 8^3, 9^2, 10^2, 11, 12\}\}.$$

The maj statistic is indeed shuffle compatible as we will show later.

The standard *q-analogue* of the nonnegative integer n is

$$[n]_q = 1 + q + q^2 + \ldots + q^{n-1}$$

where q is a variable. Given integers $0 \leq k \leq n$ the corresponding *q-binomial coefficient* is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

where

$$[n]_q! = [1]_q[2]_q \cdots [n]_q.$$

Let

$$\pi \sqcup\!\sqcup_k \sigma = \{\tau \in \pi \sqcup\!\sqcup \sigma \mid \operatorname{des}(\tau) = k\}.$$

Stanley [10], gave proofs of the identities

$$\sum_{\tau \in \pi \sqcup\!\sqcup \sigma} q^{\operatorname{maj} \tau} = q^{\operatorname{maj} \pi + \operatorname{maj} \sigma} \begin{bmatrix} |\pi| + |\sigma| \\ |\pi| \end{bmatrix}_q$$

and

$$\sum_{\tau \in \pi \sqcup\!\sqcup_k \sigma} q^{\operatorname{maj} \tau} = q^{\operatorname{maj} \pi + \operatorname{maj} \sigma + (k - \operatorname{des} \pi)(k - \operatorname{des} \sigma)} \begin{bmatrix} |\pi| - \operatorname{des}(\pi) + \operatorname{des}(\sigma) \\ k - \operatorname{des}(\pi) \end{bmatrix}_q \times \begin{bmatrix} |\sigma| - \operatorname{des}(\sigma) + \operatorname{des}(\pi) \\ k - \operatorname{des}(\sigma) \end{bmatrix}_q.$$

These imply that maj and des are shuffle compatible. He utilized P-partitions to obtain them, and later bijective proofs were given by Goulden [3] and Novick [7]. Stadler [9] also gave a bijective proof of the first identity.

A recent paper by Gessel and Zhuang [2] introduced the idea of a shuffle compatible permutation statistic and proceeded to show that many permutation statistics in fact do have this property. In addition, they showed that a descent statistic being shuffle compatible is equivalent to the existence of a certain algebra that is a quotient algebra of the Hopf algebra QSym of quasisymmetric functions. The algebra QSym can itself be identified as the shuffle algebra of the descent set statistic Des. The methods of Gessel and Zhuang were primarily algebraic using noncommutative symmetric functions, quasisymmetric functions, and variants of quasisymmetric functions to prove that statistics were shuffle compatible. They were also able to characterize many of the algebras corresponding to these statistics, and conjectured that several permutation statistics were shuffle compatible. Some of these conjectures were then proven by Grinberg in [4] using enriched P-partitions similar to those developed by Stembridge in [11]. It was also conjectured in [2] that perhaps it was true that all shuffle compatible permutation statistic are descent statistics. This has been shown to be false as an example of a shuffle compatible permutation statistic that is not a descent statistic was given by Oğuz in [8].

In this paper we present a bijective approach to showing that permutation statistics are shuffle compatible. Our method has the following three advantages. First of all, it is uniform in that essentially the same steps are followed to achieve each result. In addition, our proofs can be shorter and more transparent than other methods. Finally, we are also

able to prove shuffle compatibility for $(\mathrm{udr}, \mathrm{pk})$, one of the statistics conjectured to be shuffle compatible by Gessel and Zhuang which have resisted other techniques.

The rest of this paper is structured as follows. Section 2 gives a summary of the definitions of the various permutation statistics that we study. In Section 3 we outline our general approach to proving shuffle compatibility as well as proving one of the main reductions that is used repeatedly. Bijective proofs for the shuffle compatibility of the known shuffle compatible set valued statistics are given in Section 4. Section 5 explores shuffle compatibility of those statistics related to the major index and descent number. In Section 6 we consider those statistics related to peaks. We conclude with a section outlining possible future directions.

2. Permutation statistic definitions

Let $\pi \in L(U)$ be a permutation. Set $m = |U|$. The following is a list of all permutation statistics we will consider. We use the convention that when the name of a statistic is capitalized it is a set-valued statistic, while lower case names are used for integer valued statistics.

(i) Recall that the *descent set*, Des is defined by

$$\mathrm{Des}(\pi) = \{i \mid i \text{ is a descent of } \pi\} \subseteq [m-1]$$

and the *descent number* is the number of descents in the permutation, $\mathrm{des}(\pi) = \#\,\mathrm{Des}(\pi)$. An *ascent* of a permutation is a position i such that $\pi_i < \pi_{i+1}$. The set of the positions of ascents is denoted $\mathrm{Asc}(\pi)$, and $\mathrm{asc}(\pi)$ is the number of ascents. Two related permutations statistics are

$$\chi^-(\pi) = \begin{cases} 1 & \text{if } 1 \in \mathrm{Des}(\pi), \\ 0 & \text{if } 1 \notin \mathrm{Des}(\pi) \end{cases}$$

$$\chi^+(\pi) = \begin{cases} 1 & \text{if } m-1 \in \mathrm{Asc}(\pi), \\ 0 & \text{if } m-1 \notin \mathrm{Asc}(\pi) \end{cases}$$

For example, if $\pi = 685934$ then $\mathrm{Des}(\pi) = \{2, 4\}$ since $8 > 5$ and $9 > 3$. Therefore $\chi^-(\pi) = 0$, but $\chi^+(\pi) = 1$.

(ii) Also as previously introduced, the *major index* is given by

$$\mathrm{maj}(\pi) = \sum_{i \in \mathrm{Des}(\pi)} i.$$

(iii) A *peak* of a permutation is a position i such that $\pi_{i-1} < \pi_i > \pi_{i+1}$. The *peak set* is

$$\mathrm{Pk}(\pi) = \{i \mid \pi_{i-1} < \pi_i > \pi_{i+1}\} \subseteq [2, m-1],$$

and $\mathrm{pk}(\pi) = \#\,\mathrm{Pk}(\pi)$ is the *peak number*. A *valley* of a permutation is a position i such that $\pi_{i-1} > \pi_i < \pi_{i+1}$. The *valley set*, $\mathrm{Val}(\pi)$, and the valley number $\mathrm{val}(\pi)$ are defined analogously. Returning to our example, if $\pi = 685934$ then $\mathrm{Pk}(\pi) = \{2, 4\}$, $\mathrm{pk} = 2$, $\mathrm{Val} = \{3, 5\}$, and $\mathrm{val} = 2$.

(iv) A *left peak* is a peak of the sequence 0π, a *right peak* is a peak of $\pi 0$, and an *exterior peak* is a peak of $0\pi 0$. The initial 0 is added at position 0 and the final 0 is added at position $m + 1$. We then have the *left peak set*, Lpk, *left peak number* lpk, the *right peak set* Rpk, the *right peak number*, rpk, the *exterior peak set*, Epk, and the *exterior peak number*, epk. Continuing the previous example, $\mathrm{Lpk}(\pi) = \{2, 4\}$, $\mathrm{lpk}(\pi) = 2$, $\mathrm{Rpk}(\pi) = \mathrm{Epk}(\pi) = \{2, 4, 6\}$, and $\mathrm{rpk} = \mathrm{epk}(\pi) = 3$.

(v) A *left valley* is a valley of $\infty\pi$, a *right valley* is a valley of $\pi\infty$, and an *exterior valley* is a valley of $\infty\pi\infty$. The definitions of the following statistics for valleys are analogous to those for peaks:

$$\mathrm{Rval},\ \mathrm{Lval},\ \mathrm{Eval},\ \mathrm{rval},\ \mathrm{lval},\ \mathrm{eval}\,.$$

In our running example, $\mathrm{Rval}(\pi) = \{3, 5\}$, $\mathrm{rval}(\pi) = 2$, $\mathrm{Lval}(\pi) = \mathrm{Eval}(\pi) = \{1, 3, 5\}$, and $\mathrm{eval}(\pi) = \mathrm{lval}(\pi) = 3$.

(vi) A *monotone factor* of a permutations is a factor that is either strictly increasing or strictly decreasing. A *birun* is a maximal monotone factor. An *updown* run is a birun of 0π. The number of updown runs is denoted udr. The number of biruns itself is not shuffle compatible, but it affords the most convenient definition of udr which is. As we will see in Section 6, one can also define udr using a linear combination of pk, χ^+, and χ^-. Using the usual example, $\mathrm{udr}(\pi) = 5$, where the 5 maximal monotone factors of 0π are 068, 85, 59, 93, and 34.

3. A general approach

In this section we describe a method that is general enough to tackle most of the known shuffle compatible permutation statistics in a uniform and bijective manner. Let St be a descent statistic. In order to show it is shuffle compatible bijectively we will use the following outline.

(i) Reduce to showing only a special case of shuffle compatibility using one of the equivalences in Corollary 3.2. For the rest of this outline we assume that the equivalence of parts (1) and (2) is being used and let $m = |\pi|$.

(ii) Find a set $\Pi \subseteq L([m])$, called the set of *canonical permutations*, such that if $\pi, \pi' \in \Pi$ and the hypotheses of Definition 1.2 are satisfied with $\sigma = \sigma'$, then clearly $\mathrm{St}(\pi \sqcup\!\sqcup \sigma) = \mathrm{St}(\pi' \sqcup\!\sqcup \sigma)$.

(iii) Find a function $d : L([m]) \to \mathbb{N}$ such that for any $\pi \notin \Pi$ there is a $\pi' \in L([m])$ with $\mathrm{St}(\pi') = \mathrm{St}(\pi)$ and $d(\pi') < d(\pi)$ as well as a bijection $\pi \sqcup\!\sqcup \sigma \to \pi' \sqcup\!\sqcup \sigma$ which either preserves St or modifies it in a uniform way.

To see that this suffices to show shuffle compatibility, repeatedly apply step (iii) to generate a sequence of permutations with decreasing values of d and corresponding bijections which we assume to be St-preserving. (A similar approach will work if these bijections modify St in a uniform way.) This can only be done a finite number of times since the range of d is $\mathbb{N}$ which is well ordered. Upon termination we must have $\pi' \in \Pi$ with $\mathrm{St}(\pi') = \mathrm{St}(\pi)$ and, via composition, an St-preserving bijection $\pi \sqcup\!\sqcup \sigma \to \pi' \sqcup\!\sqcup \sigma$ where π is the permutation we started with. By step (ii), this is enough to prove shuffle compatibility. We also note that often the bijections in step (iii) will be constructed using (variations of) a map which we will call the fundamental bijection, see Definition 4.1.

The next lemma gives our first reduction. It is at the heart of the proof of Corollary 3.2 which is our main tool for reducing the number of cases under consideration. It is also a special case of the shuffle compatibility of the descent set. An example of its proof is given afterwards.

Lemma 3.1. *Let* St *be a descent statistic, and consider four permutations* $\pi, \pi', \sigma, \sigma'$ *such that* $\mathrm{gs}(\pi) \cap \mathrm{gs}(\sigma) = \mathrm{gs}(\pi') \cap \mathrm{gs}(\sigma') = \emptyset$. *If* $\mathrm{std}\,\pi = \mathrm{std}\,\pi'$ *and* $\mathrm{std}\,\sigma = \mathrm{std}\,\sigma'$ *then*

$$\mathrm{St}(\pi \sqcup\!\sqcup \sigma) = \mathrm{St}(\pi' \sqcup\!\sqcup \sigma').$$

Proof. Our method of proof will reflect the philosophy of our general approach, but with some modifications since we are only showing a special case of shuffle compatibility and do not yet have the full power of Corollary 3.2. In place of (i) above, we reduce the possible ground sets of our permutations by observing that since permutation statistics only depend on the relative order we may assume without loss of generality that

$$\mathrm{gs}(\pi) \sqcup \mathrm{gs}(\sigma) = \mathrm{gs}(\pi') \sqcup \mathrm{gs}(\sigma') = [m+n] \tag{1}$$

where $m = |\pi| = |\pi'|$, and $n = |\sigma| = |\sigma'|$. Let $U = [m]$ and $V = [n] + m$.

To mimic (ii), we consider the set

$$\Pi = L(U) \times L(V).$$

For suppose we have $(\pi, \sigma), (\pi', \sigma') \in \Pi$ satisfying the hypotheses of the Lemma. Then $\pi, \pi' \in L([m])$ implies

$$\pi = \mathrm{std}\,\pi = \mathrm{std}\,\pi' = \pi'.$$

Similarly $\sigma = \sigma'$. So clearly $\mathrm{St}(\pi \sqcup\!\sqcup \sigma) = \mathrm{St}(\pi' \sqcup\!\sqcup \sigma')$.

For (iii), we assume only (1), and produce an St-preserving bijection

$$\pi \sqcup\!\sqcup \sigma \to \mathrm{std}_U(\pi) \sqcup\!\sqcup \mathrm{std}_V(\sigma).$$

Our measure of how close a pair of permutations is to being in Π is given by $\#O$ where

$$O = \{(i,j) \in \mathrm{gs}(\pi) \times \mathrm{gs}(\sigma) \mid i > j\}.$$

A pair will have $\#O = 0$ exactly when $(\pi, \sigma) \in \Pi$.

Now if $(\pi, \sigma) \notin \Pi$ we will produce a pair of permutations (π'', σ'') with $\mathrm{St}(\pi) = \mathrm{St}(\pi'')$, $\mathrm{St}(\sigma) = \mathrm{St}(\sigma'')$ and a St-preserving bijection $\pi \sqcup\!\sqcup \sigma \to \pi'' \sqcup\!\sqcup \sigma''$ that reduces $\#O$. This suffices because repeatedly applying this operation will produce the pair of permutations $(\mathrm{std}_U(\pi), \mathrm{std}_V(\sigma)) \in \Pi$, and (by composition) the desired St-preserving bijection. An analogous argument gives a St-preserving bijection $\pi' \sqcup\!\sqcup \sigma' \to \mathrm{std}_U(\pi') \sqcup\!\sqcup \mathrm{std}_V(\sigma')$. Then we will have, using the argument in the paragraph about part (ii),

$$\mathrm{St}(\pi \sqcup\!\sqcup \sigma) = \mathrm{St}(\mathrm{std}_U(\pi) \sqcup\!\sqcup \mathrm{std}_V(\sigma)) = \mathrm{St}(\mathrm{std}_U(\pi') \sqcup\!\sqcup \mathrm{std}_V(\sigma')) = \mathrm{St}(\pi' \sqcup\!\sqcup \sigma')$$

as required.

We now construct (π'', σ'') and the St-preserving bijection. Since $(\pi, \sigma) \notin \Pi$, there exists a pair $(i, i-1) \in O$ such that $i \in \mathrm{gs}(\pi)$ and $i - 1 \in \mathrm{gs}(\sigma)$. Set $\pi'' = (i, i-1)\pi$ and $\sigma'' = (i, i-1)\sigma$ where $(i, i-1)\pi$ is the permutation π with i replaced by $i-1$ and similarly for $(i, i-1)\sigma$. Let $\tau \in \pi \sqcup\!\sqcup \sigma$. Then the bijection is given by

$$T_i(\tau) = \begin{cases} (i, i-1)\tau & \text{if } i,\ i-1 \text{ are not adjacent in } \tau, \\ \tau & \text{otherwise,} \end{cases}$$

where $(i, i-1)\tau$ is τ with i and $i-1$ interchanged.

This map is its own inverse, hence a bijection. To see that the image of the map is in $\pi'' \sqcup\!\sqcup \sigma''$ note that if $i, i-1$ are not adjacent then $T_i(\tau) \in \pi'' \sqcup\!\sqcup \sigma''$ since $T_i(\tau)$ is the unique shuffle of π'' and σ'' whose word satisfies $\omega(T_i(\tau)) = \omega(\tau)$. And if i and $i-1$ are adjacent in τ, then τ is easily seen to also be a shuffle of π'' and σ''.

The map T_i is Des preserving because swapping the positions of $i, i-1$ when i and $i-1$ are not adjacent will not change the order relation between any adjacent pairs. Indeed, given any $j \notin \{i-1, i\}$ that is adjacent to one or both of $i, i-1$, it is clear that either both $j > i$ and $j > i-1$, or $j < i-1$ and $j < i$, and hence the inequalities are preserved when interchanging $i, i-1$. It follows from the definition of a descent statistic that T_i is also St preserving. □

As an example, let $\boldsymbol{U} = \{\mathbf{1}, \mathbf{2}, \mathbf{4}\}$, $V = \{3, 7\}$ and $\boldsymbol{\pi} = \mathbf{241}$ and $\sigma = 73$. Then

$$\pi \sqcup\!\sqcup \sigma = \{\mathbf{24}173,\ \mathbf{24}713,\ \mathbf{24}731,\ \mathbf{27}413,\ \mathbf{27}431,\ \mathbf{27}341,\ 7\mathbf{24}13,\ 7\mathbf{24}31,\ 7\mathbf{23}41,\ 73\mathbf{241}\}.$$

Taking St to be the peak set statistic,

$$\mathrm{Pk}(\pi \sqcup\!\sqcup \sigma) = \{\{\{2\}^2,\ \{3\}^4,\ \{4\}^2,\ \{2,4\}^2\}\}.$$

Now standardize to $[m+n]$ by replacing σ with $\widetilde{\sigma} = 53$, and V with $\widetilde{V} = \{3, 5\}$ to obtain

$$\pi \sqcup\!\sqcup \widetilde{\sigma} = \{\mathbf{24}153,\ \mathbf{24}513,\ \mathbf{24}531,\ \mathbf{25}413,\ \mathbf{25}431,\ \mathbf{25}341,\ 5\mathbf{24}13,\ 5\mathbf{24}31,\ 5\mathbf{23}41,\ 53\mathbf{24}1\}.$$

Clearly $\mathrm{Pk}(\pi \sqcup\!\sqcup \widetilde{\sigma}) = \mathrm{Pk}(\pi \sqcup\!\sqcup \sigma)$. We next would like to change $\boldsymbol{U}$ to $\boldsymbol{U'} = \{\mathbf{1,2,3}\}$ and $\widetilde{V}$ to $\widetilde{V}' = \{4, 5\}$. This can be done using $(4, 3) \in O$. We apply

$$T_4(\pi \sqcup\!\sqcup \widetilde{\sigma}) = \{\mathbf{23}154, \mathbf{23}514, \mathbf{23}541, \mathbf{25}314, \mathbf{25}431, \mathbf{25}341, 5\mathbf{23}14, 5\mathbf{24}\mathbf{31}, 5\mathbf{23}41, 54\mathbf{23}1\}$$

where, for instance, $T_4(5\mathbf{24}13) = 5\mathbf{23}14$ since 3 and 4 are not adjacent. On the other hand, $T_4(5\mathbf{23}\mathbf{41}) = 5\mathbf{23}\mathbf{41}$ since $3, 4$ are adjacent. One can check that the distribution with respect to Pk remains unchanged.

The following corollary shows that in order to check shuffle compatibility, it suffices to check the special case when the ground sets of the permutation have some fixed relation with each other. This reduction greatly simplifies the required arguments for showing statistics are shuffle compatible.

Corollary 3.2. *Suppose* St *is a descent statistic. The following are equivalent.*

(1) *The statistic* St *is shuffle compatible.*
(2) *If* $\mathrm{St}(\pi) = \mathrm{St}(\pi')$ *where* $\pi, \pi' \in L([m])$*, and* $\sigma \in L([n] + m)$ *for some* $m, n \geq 0$*, then*

$$\mathrm{St}(\pi \sqcup\!\sqcup \sigma) = \mathrm{St}(\pi' \sqcup\!\sqcup \sigma).$$

(3) *If* $\mathrm{St}(\sigma) = \mathrm{St}(\sigma')$ *where* $\sigma, \sigma' \in L([n] + m)$*, and* $\pi \in L([m])$ *for some* $m, n \geq 0$*, then*

$$\mathrm{St}(\pi \sqcup\!\sqcup \sigma') = \mathrm{St}(\pi \sqcup\!\sqcup \sigma).$$

Proof. It is clear that (1) implies (2) and (3) as these are special cases of the definition of shuffle compatibility. Assume (2) holds and let π, π' be any two permutations of the same length m such that $\mathrm{St}(\pi) = \mathrm{St}(\pi')$. Let σ, σ' be two permutations of the same length n and disjoint from π, π' (respectively) such that $\mathrm{St}(\sigma) = \mathrm{St}(\sigma')$. Set $U = [m]$, $U^+ = [m] + n$, $V = [n]$, and $V^+ = [n] + m$. Then by Lemma 3.1 and our assumption,

$$\begin{aligned}
\mathrm{St}(\pi \sqcup\!\sqcup \sigma) &= \mathrm{St}(\mathrm{std}_U(\pi) \sqcup\!\sqcup \mathrm{std}_{V^+}(\sigma)) && \text{by Lemma 3.1} \\
&= \mathrm{St}(\mathrm{std}_U(\pi') \sqcup\!\sqcup \mathrm{std}_{V^+}(\sigma)) && \text{by (2)} \\
&= \mathrm{St}(\mathrm{std}_{U^+}(\pi') \sqcup\!\sqcup \mathrm{std}_V(\sigma)) && \text{by Lemma 3.1} \\
&= \mathrm{St}(\mathrm{std}_{U^+}(\pi') \sqcup\!\sqcup \mathrm{std}_V(\sigma')) && \text{by (2)} \\
&= \mathrm{St}(\pi' \sqcup\!\sqcup \sigma') && \text{by Lemma 3.1.}
\end{aligned}$$

The proof that (3) implies (1) is very similar. □

4. Set valued statistics

Our first main results are to give bijective proofs for the shuffle compatibility of some set valued statistics. The statistic Des was given a different bijective proof in [2], so the novelty here is the bijective proofs of the remaining set valued statistics as well as the uniform manner in which they are attained.

Definition 4.1. Given permutations π, σ with disjoint ground sets and a third permutation π' disjoint from σ with $|\pi| = |\pi'|$, define the *fundamental bijection*

$$\Phi : \pi \sqcup\!\sqcup \sigma \to \pi' \sqcup\!\sqcup \sigma$$

by

$$\Phi(\tau) = \tau'$$

where $\tau' \in \pi' \sqcup\!\sqcup \sigma$ is the unique permutation with $\omega(\tau') = \omega(\tau)$. This amounts to replacing the elements of π with the elements of π' in the same order and positions as in τ. If instead one holds π fixed and replaces σ with σ' then one obtains a bijection which we call $\widetilde{\Phi}$. □

For example, if $\pi = 132$, $\sigma = 4589$, $\tau = 1453829 \in \pi \sqcup\!\sqcup \sigma$ and $\pi' = 361$, then $\Phi(\tau) = 3456819$.

The following theorem establishes the shuffle compatibility of some set valued statistics. These were proven in this language by Gessel, Zhuang and Grinberg in [2] and [4] by lengthier and primarily algebraic methods. We should also mention that the shuffle compatibility of Des, Pk, and Lpk follows from earlier work of Stanley [10], Stembridge [11], and Petersen [5,6], respectively An advantage of our approach is the directness and uniformity with which the results are obtained.

Theorem 4.2. *The set valued statistics* Des*,* Pk*,* Lpk*,* Rpk*, and* Epk *are all shuffle compatible.*

Proof. By Corollary 3.2, part (2), we may reduce to showing that for $\pi, \pi' \in L([m])$ and $\sigma \in L([n] + m)$, if we have $\mathrm{St}(\pi) = \mathrm{St}(\pi')$ then it follows that $\mathrm{St}(\pi \sqcup\!\sqcup \sigma) = \mathrm{St}(\pi' \sqcup\!\sqcup \sigma)$ for the five statistics listed above.

The fundamental bijection $\Phi : \pi \sqcup\!\sqcup \sigma \to \pi' \sqcup\!\sqcup \sigma$ will be the map we will use for all five statistics. Because these cases are so straightforward, we will not have to find a canonical set of permutations.

For each of these statistics, $\mathrm{St} \in \{\mathrm{Des}, \mathrm{Pk}, \mathrm{Lpk}, \mathrm{Rpk}, \mathrm{Epk}\}$, we will give a complete list of the cases that determine whether a given position will contribute to $\mathrm{St}(\tau)$ for a shuffle $\tau \in \pi \sqcup\!\sqcup \sigma$. It will then be easy to check that St will be preserved by Φ.

- **Descent set,** Des:
  Observe that in a shuffle $\tau \in \pi \sqcup\!\sqcup \sigma$ we have $i \in \operatorname{Des}\tau$ if and only if $\tau_i\tau_{i+1}$ equals one of
  1. $\pi_j\pi_{j+1}$ where $j \in \operatorname{Des}\pi$,
  2. $\sigma_k\sigma_{k+1}$ where $k \in \operatorname{Des}\sigma$, or
  3. $\sigma_k\pi_j$.

  It is now easy to check that the descent set is preserved in passing from τ to $\Phi(\tau)$.
- **Peak set,** Pk:
  For a shuffle $\tau \in \pi \sqcup\!\sqcup \sigma$, we have $i \in \operatorname{Pk}\tau$ if and only if $\tau_{i-1}\tau_i\tau_{i+1}$ equals one of
  1. $\pi_{j-1}\pi_j\pi_{j+1}$ where $j \in \operatorname{Pk}\pi$,
  2. $\sigma_{k-1}\sigma_k\sigma_{k+1}$ where $k \in \operatorname{Pk}\sigma$,
  3. $\sigma_k\sigma_{k+1}\pi_j$ where $k \in \operatorname{Asc}\sigma$,
  4. $\pi_j\sigma_k\sigma_{k+1}$ where $k \in \operatorname{Des}\sigma$,
  5. $\pi_j\sigma_k\pi_{j+1}$.

  This makes it simple to check that a peak in τ will remain one in $\Phi(\tau)$. For example, in case 3 we have $\sigma_k < \sigma_{k+1} > \pi_j$ in τ so that the position of σ_{k+1} is a peak of τ. Upon replacing π with π' we have $\sigma_k < \sigma_{k+1} > \pi'_j$ since every element of π' is less than every element of σ. Therefore the position of σ_{k+1} is a peak in $\Phi(\tau)$ at the same position as it was in τ. Using similar arguments and Φ^{-1}, one sees that a position that is a peak of $\Phi(\tau)$ must also be a peak of τ and so Φ is peak preserving.
- **Left peak set,** Lpk:
  For a shuffle $\tau \in \pi \sqcup\!\sqcup \sigma$, note that we have $\operatorname{Pk}(\tau) \subseteq \operatorname{Lpk}(\tau)$ so that the above cases for the peak set show that for $i \geq 2$ we have $i \in \operatorname{Lpk}(\tau)$ if and only if $i \in \operatorname{Lpk}(\Phi(\tau))$. It therefore remains only to check what happens when $i = 1$. But $1 \in \operatorname{Lpk}(\tau)$ if and only if $0\tau_1\tau_2$ equals one of
  1. $0\pi_1\pi_2$ where $1 \in \operatorname{Lpk}(\pi)$
  2. $0\sigma_1\sigma_2$ where $1 \in \operatorname{Lpk}(\sigma)$
  3. $0\sigma_1\pi_1$

  The check that left peaks at $i = 1$ are preserved is similar to the arguments for Pk. So here and for the following statistics we have left this verification to the reader.
- **Right peak set,** Rpk:
  The argument is analogous to that of Lpk, except that we now need additional cases at the right end of τ. Note that $m + n \in \operatorname{Rpk}(\tau)$ if and only if $\tau_{m+n-1}\tau_{m+n}0$ equals one of
  1. $\pi_{m-1}\pi_m0$ where $m \in \operatorname{Rpk}(\pi)$
  2. $\sigma_{n-1}\sigma_n0$ where $n \in \operatorname{Rpk}(\sigma)$
  3. $\pi_m\sigma_n0$
- **External peak set,** Epk:
  Since $\operatorname{Epk}(\tau) = \operatorname{Lpk}(\tau) \cup \operatorname{Rpk}(\tau)$ and the single bijection Φ preserves both Rpk and Lpk, we have that Epk is also preserved under Φ.

This completes the proof of the shuffle compatibility of these five statistics. □

Note that if there is a single bijection which shows the shuffle compatibility of two or more permutation statistics, then it follows immediately that any tuple of these statistics is also shuffle compatible. For example, since Lpk and Rpk are both shuffle compatible by means of the bijection Φ, then so is $(\mathrm{Lpk}, \mathrm{Rpk})$. Therefore any tuple from the above five statistics is also shuffle compatible. This gives one answer to a question of Gessel and Zhuang in [2] as to when a tuple of statistics is shuffle compatible.

Gessel and Zhuang also point out [2, Lemma 3.6] that one can get the shuffle compatibility of valley statistics from those for the corresponding peak statistics by simple bijections. So here we merely state the valley equivalent of the previous theorem.

Corollary 4.3. *The statistics* Asc, Val, Lval, Rval, *and* Eval *are shuffle compatible.*

5. The major index

For the next proof, we need to introduce a labeling on the spaces of a permutation. Let π be a permutation of length m with $\mathrm{des}(\pi) = k$. Then by a *space* of π we mean the gap between two adjacent elements of π. There is, by convention, an initial space before the first element of π and a final space after the last element of the permutation. Label these spaces by assigning the right-most (final) space the label 0 then labeling the spaces after descents of π with the integers in $[k]$ from right to left, then labeling the remaining spaces with the integers in $[k+1, m]$ from left to right. Equivalently, we label the spaces of π corresponding to descents of $0\pi0$ from right to left, and then the spaces of π corresponding to ascents of $0\pi0$ from left to right using the elements of $[0, m]$. In what follows we make no distinction between a space and its label. For example if $\pi = 265781$ then the labeled permutation is

$$^{3}2^{4}6^{2}5^{5}7^{6}8^{1}1^{0}$$

with the raised numbers being the labels of the spaces. If π_i and π_{i+1} are the elements on either side of space x then we say there is a *descent* or *ascent at space* x if $i \in \mathrm{Des}(\pi)$ or $i \in \mathrm{Asc}(\pi)$, respectively.

It is well known that inserting a number greater than $\max \mathrm{gs}(\pi)$ in space i increases $\mathrm{maj}\,\pi$ by i. Continuing our example, inserting 9 in space 4 of π gives the permutation 2965781 with $\mathrm{maj}(2965781) = 11 = \mathrm{maj}(265781) + 4$. This fact is used in one of the standard proofs that the generating function for maj over the permutations of $[n]$ is $[n]_q!$. We will now see that this is a crucial tool for proving certain shuffle compatibility results.

Theorem 5.1. *The permutation statistics* des *and* $(\mathrm{maj}, \mathrm{des})$ *are shuffle compatible.*

Proof. Our first step is to use Corollary 3.2 part (3) to reduce to showing that $\pi \in L(m)$ and $\sigma, \sigma' \in L([n] + m)$ with $\mathrm{St}(\sigma) = \mathrm{St}(\sigma')$ implies $\mathrm{St}(\pi \sqcup\!\sqcup \sigma) = \mathrm{St}(\pi \sqcup\!\sqcup \sigma')$ for each

$\mathrm{St} \in \{\mathrm{des}, (\mathrm{maj}, \mathrm{des})\}$. The core of the proof is the existence of certain bijections that preserve des, lower maj by one, and allow us to replace σ with a permutation that is closer to being in our chosen set of canonical permutations, as outlined in the general approach. For the permutations with $\mathrm{des}\,\sigma = p$ we will use the canonical set

$$\Pi = \{\sigma \in L([n] + m) \mid \mathrm{Des}(\sigma) = [p]\}$$

which consists of the permutations with a sequence of p descents followed by a sequence of ascents. Given two permutations $\sigma, \sigma' \in \Pi$, we know by Theorem 4.2 that $\mathrm{Des}(\pi \sqcup\!\sqcup \sigma) = \mathrm{Des}(\pi \sqcup \sigma')$ and hence the same holds for any descent statistic. This shows that part (ii) of the general approach is satisfied.

Our measure of how close a permutation is to being in Π is $d : L([n]+m) \to \mathbb{N}$ given by $d(\sigma) = \mathrm{maj}(\sigma)$. Note that among all permutations in $L([n] + m)$ with $\mathrm{des}\,\sigma = p$, those in Π have the minimum possible maj, namely $\binom{p+1}{2}$. Our strategy will be to find a bijection between shuffles $\tau \in \pi \sqcup \sigma$ and the shuffles of π with an element of Π which preserves des and lowers the major index of each τ by the same amount, namely $\mathrm{maj}(\sigma) - \binom{p+1}{2}$. This will prove the theorem.

To reduce a permutation to one in Π we will move its descents to the left one position at a time. More specifically, if $\sigma \notin \Pi$ then there is at least one position $i \geq 2$ such that $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$. Let σ'' be any permutation such that

$$\mathrm{Des}(\sigma'') = (\mathrm{Des}(\sigma) \setminus \{i\}) \cup \{i - 1\}.$$

Note that this preserves des but lowers maj by one in passing from σ to σ''. The bijection we will define between $\pi \sqcup\!\sqcup \sigma$ and $\pi \sqcup\!\sqcup \sigma''$ will have the same properties and so, by iteration, complete the proof.

For $\tau \in \pi \sqcup \sigma$, write τ as a concatenation $\tau = \tau^a \tau^b \tau^c$ where τ^b is the factor of τ between but not including σ_{i-1} and σ_{i+1}. Then τ^a and τ^c are the remaining initial and final factors of τ, respectively. Note that there is exactly one element of σ in τ^b and that it is larger than all the elements of π. Consider the permutation δ that is τ^b with σ_i removed. All spaces will be spaces of δ. Let x be the space of δ from which σ_i was removed and set $(\tau^b)''$ to be the permutation δ with σ_i inserted into the space $x - 1$ where $x - 1$ is taken modulo $|\delta| + 1$.

Define a map $\Theta : \pi \sqcup \sigma \to \pi \sqcup \sigma''$ by

$$\Theta(\tau) = \tau''$$

where τ'' is the unique element of $\pi \sqcup \sigma''$ such that $\omega(\tau'') = \omega(\tau^a (\tau^b)'' \tau^c)$.

We now show that Θ has the desired properties, namely that it is des preserving and satisfies $\mathrm{maj}(\Theta(\tau)) = \mathrm{maj}(\tau) - 1$. There are two cases to check, based on the label x of the original space that σ_i occupied.

1. If $1 \leq x \leq \mathrm{des}(\delta) + 1$:

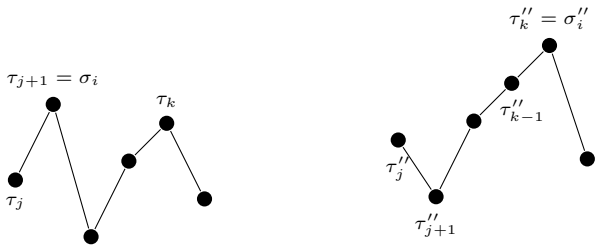

Fig. 1. A schematic drawing for the case when $1 \leq x \leq \mathrm{des}(\delta) + 1$ and $j + 2 \neq k$.

First note that in this case δ cannot be empty since x exists in this range. Let τ_j be the element of τ directly before σ_i and let τ_k be the element of τ directly before space $x - 1$ of δ.

The map Θ removes σ_i from the position after τ_j and inserts σ''_i in the position directly after τ_k while changing the order relation from $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$ to $\sigma''_{i-1} > \sigma''_i < \sigma''_{i+1}$. There are no descents, l, in τ with $j + 2 \leq l \leq k - 1$. Therefore we only have to analyze what happens at positions $j,\ j+1,\ k-1$, and k. First, assume that $j + 2 \neq k$.

An illustration of this case in a generic small example is shown in Fig. 1. The left picture is an example of the initial state of τ^b and the right picture is of the resulting part of τ'' after applying Θ. Each node represents an element of τ or τ'' and the lines connecting them represent the order relation between adjacent elements. For example, a line with positive slope corresponds to the first element being smaller than the second.

- In τ:
  - $j \notin \mathrm{Des}(\tau)$:
    The position j is never a descent of τ since $\tau_{j+1} = \sigma_i$, and τ_j is either σ_{i-1} or in π. In both cases $\tau_j < \tau_{j+1}$.
  - $j + 1 \in \mathrm{Des}(\tau)$:
    The position $j+1$ is always a descent of τ since $\tau_{j+1} = \sigma_i$ and $\tau_{j+2} \in \pi$ because of the range of x.
  - $k - 1 \notin \mathrm{Des}(\tau)$.
    Since $j + 2 \neq k$, the definition of the space labeling and the range of x show that $k - 1$ is an ascent of τ.
  - $k \in \mathrm{Des}(\tau)$ if and only if $x \neq 1$.
    If $x = 1$, then $\tau_k \in \pi$ and $\tau_{k+1} = \sigma_{i+1}$ and so k is an ascent of τ. On the other hand, if $x \neq 1$, then k is a descent of τ by the definition of the space labeling and the range of x.
- In τ'':
  - $j \in \mathrm{Des}(\tau'')$:
    The position j is always a descent of τ'' since $\tau''_{j+1} \in \pi$, and either $\tau''_j = \sigma''_{i-1}$ or $\tau''_j \in \pi$ with j corresponding to the descent at space x of δ.
  - $j + 1 \notin \mathrm{Des}(\tau'')$:

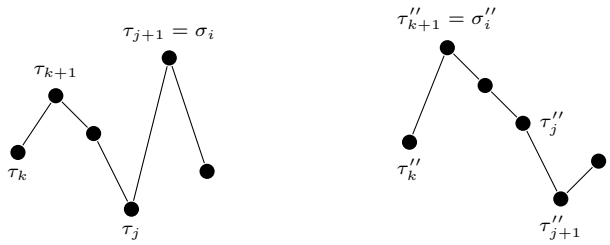

Fig. 2. A schematic drawing for the case when $\mathrm{des}(\delta) + 2 \leq x \leq |\delta| + 1$ modulo $|\delta| + 1$.

The position $j+1$ is never a descent of τ'' since $\tau''_{j+1} \in \pi$, and either $\tau''_{j+2} = \sigma''_i$ or $\tau''_{j+2} \in \pi$ with $j+1$ corresponding to an ascent of δ by the definition of the space labeling and the range of x.

- $k - 1 \notin \mathrm{Des}(\tau'')$
  Since $j + 2 \neq k$ we have $\tau''_{k-1} \in \pi$ and $\tau''_k = \sigma''_i$ so that $k-1$ is an ascent.
- $k \in \mathrm{Des}(\tau'')$ if and only if $x \neq 1$.
  We have $\tau''_k = \sigma''_i$. If $x = 1$ then $\tau''_{k+1} = \sigma''_{i+1}$ and hence $k \notin \mathrm{Des}(\tau'')$ by the choice of σ''. On the other hand if $x \neq 1$, then $\tau''_{k+1} \in \pi$ and hence $\sigma''_i > \tau''_{k+1}$.

When $j + 2 = k$, we have the same two lists but with the item concerning $k - 1$ removed since $k - 1 = j + 1$ and so the item concerning $j + 1$ covers this case.
Comparing the two lists, we have

$$\mathrm{Des}(\tau'') = (\mathrm{Des}(\tau) \setminus \{j+1\}) \cup \{j\} \tag{2}$$

and, in particular, $\mathrm{des}(\tau'') = \mathrm{des}(\tau)$. This relation between descent sets also makes it clear that $\mathrm{maj}(\tau'') = \mathrm{maj}(\tau) - 1$.

2. If $\mathrm{des}(\delta) + 2 \leq x \leq |\delta| + 1$ modulo $|\delta| + 1$:
   Define τ_j and τ_k as before.
   Again, the map Θ removes σ_i from the position after τ_j and inserts σ''_i in the position directly after τ_k while changing the order relation from $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$ to $\sigma''_{i-1} > \sigma''_i < \sigma''_{i+1}$. Note however, that it is now possible for δ to be empty. In that case Θ just changes the peak at σ_i at position $j+1$ in τ to a valley in τ''. So clearly equation (2) still holds.
   Therefore assume $\delta \neq \emptyset$ and note that all positions strictly between k and j will be descents in τ. It is easy to see that these positions will remain descents after applying Θ. So we only need to check what happens at positions $j, j+1$ and k.
   An illustration of this case is given in Fig. 2. The left picture is an illustration of an example of the initial state of τ^b and the right picture is of the resulting part τ'' after applying Θ.
   - In τ:
     - $j \notin \mathrm{Des}(\tau)$:
       The position j is never a descent of τ since $\tau_j \in \pi$ and $\tau_{j+1} = \sigma_i$ by definition.
     - $j + 1 \in \mathrm{Des}(\tau)$:

The position $j+1$ is always a descent of τ since $\tau_{j+1} = \sigma_i$ and τ_{j+2} is σ_{i+1} or is in π. In both cases $\tau_{j+1} > \tau_{j+2}$.

– $k \in \mathrm{Des}(\tau)$ if and only if $x = \mathrm{des}(\delta) + 2$.
If $x = \mathrm{des}(\delta)+2$, then $x-1 = \mathrm{des}(\delta)+1$ is the space before δ. Hence $\tau_k = \sigma_{i-1}$ and $\tau_{k+1} \in \pi$, so $k \in \mathrm{Des}(\tau)$. On the other hand, if $x \neq \mathrm{des}(\delta)+2$, then k is an ascent of τ by the definition of the space labeling and the range of x.

- In τ'':
  – $j \in \mathrm{Des}(\tau'')$:
  The position j is always a descent of τ'' since $\tau''_{j+1} \in \pi$, and either $\tau''_j = \sigma''_i$ or $\tau''_j \in \pi$ and j corresponds to the descent of δ at the space previous to x.
  – $j+1 \notin \mathrm{Des}(\tau'')$:
  The position $j+1$ is never a descent of τ'' since $\tau''_{j+1} \in \pi$, and either $\tau''_{j+2} = \sigma''_{i+1}$ or $\tau''_{j+2} \in \pi$ and $j+1$ corresponds to the ascent of δ at space x.
  – $k \in \mathrm{Des}(\tau'')$ if and only if $x = \mathrm{des}(\delta) + 2$.
  If $x = \mathrm{des}(\delta)+2$, then $x-1 = \mathrm{des}(\delta)+1$ is the space before δ. Hence $\tau''_k = \sigma''_{i-1}$ and $\tau''_{k+1} = \sigma_i$, so $k \in \mathrm{Des}(\tau)$. On the other hand, if $x \neq \mathrm{des}(\delta)+2$ then $\tau''_k \in \pi$ and $\tau''_{k+1} = \sigma''_i$, so that $k \notin \mathrm{Des}(\tau'')$.

Thus in this case as well equation (2) continues to hold.

This finishes the proof of the shuffle compatibility of des and $(\mathrm{maj}, \mathrm{des})$. □

In the above proof we proceeded to reduce permutations by moving descents to the left as far as we could. However, to show the shuffle compatibility of maj we must reduce the permutations even further since permutations with the same value of maj may have different numbers of descents.

Theorem 5.2. *The permutation statistic* maj *is shuffle compatible.*

Proof. As in the previous proof, our first step is to use Corollary 3.2 to reduce to showing that $\pi \in L(m)$ and $\sigma, \sigma' \in L([n]+m)$ with $\mathrm{maj}(\sigma) = \mathrm{maj}(\sigma')$ implies $\mathrm{maj}(\pi \sqcup\!\sqcup \sigma) = \mathrm{maj}(\pi \sqcup\!\sqcup \sigma')$. We will use the same canonical permutation for every element of $L([n]+m)$ by letting

$$\Pi = \{\sigma \in L([n]+m) \mid \mathrm{Des}(\sigma) = \emptyset\}.$$

This set contains the unique increasing permutation. Our measure of how close a permutation is to being in Π is $d : L([n]+m) \to \mathbb{N}$ given by $d(\sigma) = \mathrm{maj}(\sigma)$. Observe that $\sigma \in \Pi$ if and only if $\mathrm{maj}(\sigma) = 0$. Our strategy will be to find a bijection between shuffles $\tau \in \pi \sqcup\!\sqcup \sigma$ and shuffles with the element of Π which lowers the major index of each τ by the same amount, namely $\mathrm{maj}(\sigma)$.

To reduce a permutation to one in Π we will move their descents to the left one position at a time until they are moved to position 0 at which point they vanish. More precisely,

if $\sigma \notin \Pi$ then there exists at least one position $i \in \mathrm{Des}(\sigma)$ such that $i-1 \notin \mathrm{Des}(\sigma)$ and we allow the case $i = 1$. Fix any such i and let σ'' be any permutation such that

$$\mathrm{Des}(\sigma'') = \begin{cases} (\mathrm{Des}(\sigma) \setminus \{i\}) \cup \{i-1\} & \text{if } i \geq 2, \\ \mathrm{Des}(\sigma) \setminus \{1\} & \text{if } i = 1. \end{cases}$$

The map Θ from the proof of Theorem 5.1 suffices when $i \geq 2$, so we need only give a bijection $\widetilde{\Theta} : \pi \sqcup\!\sqcup \sigma \to \pi \sqcup\!\sqcup \sigma''$ for the case $i = 1$ such that the image $\tau'' = \widetilde{\Theta}(\tau)$ satisfies

$$\mathrm{maj}(\tau'') = \mathrm{maj}(\tau) - 1. \tag{3}$$

This means that if there is a descent at position 1 then we need a bijection which reduces maj by one by changing that descent to an ascent. Set

$$\widetilde{\sigma} = m+1, \sigma_1 + 1, \sigma_2 + 1, \ldots, \sigma_n + 1 \in L([n+1] + m)$$

and

$$\widetilde{\sigma}'' = m+n+1, \sigma_1'', \sigma_2'', \ldots, \sigma_n'' \in L([n+1] + m).$$

For a permutation σ, let

$$S_\sigma = \{\tau \in \pi \sqcup\!\sqcup \sigma \mid \tau_1 = \sigma_1\} \tag{4}$$

be the subset of $\pi \sqcup\!\sqcup \sigma$ whose elements all have σ_1 in the first position. There is a natural bijection $\iota : \pi \sqcup\!\sqcup \sigma \to S_{\widetilde{\sigma}}$. Namely, for $\tau \in \pi \sqcup\!\sqcup \sigma$, let $\iota(\tau) = (m+1)\widetilde{\tau}$ where $\widetilde{\tau}$ is the unique permutation such that $\omega(\tau) = \omega(\widetilde{\tau})$, i.e., $\widetilde{\tau}$ is $\pi \sqcup\!\sqcup \sigma$ with all elements of σ increased by 1. There is an analogous map $\iota'' : \pi \sqcup\!\sqcup \sigma'' \to S_{\widetilde{\sigma}''}$.

Note that $\Theta(S_{\widetilde{\sigma}}) = S_{\widetilde{\sigma}''}$ since

$$\mathrm{Des}(\widetilde{\sigma}'') = (\mathrm{Des}(\widetilde{\sigma}) \setminus \{2\}) \cup \{1\}.$$

It follows that we can define $\widetilde{\Theta}$ by insisting that the following diagram commutes

$$\begin{array}{ccccc} \pi \sqcup\!\sqcup \sigma & \xrightarrow{\quad\iota\quad} & S_{\widetilde{\sigma}} & \subseteq & \pi \sqcup\!\sqcup \widetilde{\sigma} \\ \Big\downarrow{\scriptstyle \widetilde{\Theta}} & & \Big\downarrow{\scriptstyle \Theta} & & \\ \pi \sqcup\!\sqcup \sigma'' & \xrightarrow{\quad\iota''\quad} & S_{\widetilde{\sigma}''} & \subseteq & \pi \sqcup\!\sqcup \widetilde{\sigma}'' \end{array}$$

In other words, we define $\widetilde{\Theta} = \iota''^{-1} \circ \Theta \circ \iota$ which is clearly bijective.

To finish, it suffices to show that $\widetilde{\Theta}$ reduces maj by 1. First of all, observe that we have

$$\operatorname{maj}(\iota(\tau)) = \begin{cases} \operatorname{maj}(\tau) + \operatorname{des}(\tau) & \text{if } \tau_1 = \sigma_1 \\ \operatorname{maj}(\tau) + \operatorname{des}(\tau) + 1 & \text{if } \tau_1 = \pi_1 \end{cases}$$

since each descent is shifted to the right by one position, and if $\tau_1 = \pi_1$ there is an additional descent at position 1. By the previous theorem, $\operatorname{maj}(\Theta(\iota(\tau))) = \operatorname{maj}(\iota(\tau)) - 1$. Also

$$\operatorname{des}(\Theta(\iota(\tau))) = \operatorname{des}(\iota(\tau)) = \begin{cases} \operatorname{des}(\tau) & \text{if } \tau_1 = \sigma_1 \\ \operatorname{des}(\tau) + 1 & \text{if } \tau_1 = \pi_1 \end{cases}$$

since, by the previous theorem, Θ was des preserving. Finally, for an element $\tau'' \in \pi \sqcup\!\sqcup \widetilde{\sigma''}$ we have

$$\operatorname{maj}(\iota''^{-1}(\tau'')) = \operatorname{maj}(\tau'') - \operatorname{des}(\tau'')$$

since each descent of τ'' is moved to the left by one position in $\iota''^{-1}(\tau'')$.

Thus, regardless of the first element of τ,

$$\begin{aligned} \operatorname{maj}(\iota''^{-1}(\Theta(\iota(\tau)))) &= \operatorname{maj}(\Theta(\iota(\tau)) - \operatorname{des}(\Theta(\iota(\tau)) \\ &= (\operatorname{maj}(\iota(\tau)) - 1) - \operatorname{des}(\iota(\tau)) \\ &= \operatorname{maj}(\tau) - 1, \end{aligned}$$

which finishes the proof. □

We can now recover the identity for the distribution of maj over the shuffle set. We start with a well-known result whose proof can be found in [1] Section 2.2.2. Bóna's treatment deals with rearrangements of a multiset containing ones and twos, which is easily seen to be equivalent to shuffling two words, the first consisting only of ones and the second only of twos. One defines maj in the same way for words with repeated numbers.

Theorem 5.3. *Let* $e = \overbrace{11\ldots1}^{m}$ *and* $f = \overbrace{22\ldots2}^{n}$ *be words of length* m *and* n *respectively. Then*

$$\sum_{\tau \in e \sqcup\!\sqcup f} q^{\operatorname{maj}(\tau)} = \begin{bmatrix} m+n \\ m \end{bmatrix}_q. \tag{5}$$

The previous result will act as the base case for our inductive proof of the following result cited in the introduction.

Theorem 5.4. *Let* π *and* σ *be permutations with disjoint ground sets and lengths* m *and* n*, respectively. Then*

$$\sum_{\tau \in \pi \sqcup\!\sqcup \sigma} q^{\operatorname{maj} \tau} = q^{\operatorname{maj} \pi + \operatorname{maj} \sigma} \begin{bmatrix} m+n \\ m \end{bmatrix}_q .$$

Proof. Since we have shown that maj is shuffle compatible, we may assume that $\pi \in L([m])$ and $\sigma \in L([n]+m)$. We will induct on $\operatorname{maj}(\pi) + \operatorname{maj}(\sigma)$. If $\operatorname{maj}(\pi) + \operatorname{maj}(\sigma) = 0$ then $\pi = 12 \dots m$ and $\sigma = m+1, m+2, \dots, m+n$. In this case, the result follows from Theorem 5.3. This is because replacing e with π and f with σ, respectively, in $\tau \in e \sqcup\!\sqcup f$ turns a repeated pair 11 or 22 into an ascent, while descents remain descents since all elements of σ are larger than those of π.

Now assume $\operatorname{maj}(\pi) + \operatorname{maj}(\sigma) > 0$. By Lemma 3.1 we can assume, without loss of generality, that $\operatorname{maj}(\sigma) > 0$. So the map $\widetilde{\Theta} : \pi \sqcup\!\sqcup \sigma \to \pi \sqcup\!\sqcup \sigma''$ of Theorem 5.2 is a bijection, where $\operatorname{maj}(\sigma'') = \operatorname{maj}(\sigma) - 1$ and $\operatorname{maj}(\tau'') = \operatorname{maj}(\tau) - 1$ for $\tau'' = \widetilde{\Theta}(\tau)$. By induction, the desired equation holds for $\pi \sqcup\!\sqcup \sigma''$. Multiplying the equality by q and substituting, shows that it also holds for $\pi \sqcup\!\sqcup \sigma$. □

6. Peak statistics

We now move on to statistics related to peaks. The proof for the statistic $(\operatorname{udr}, \operatorname{pk})$ is notable because it was previously only conjectured to be shuffle compatible and here we give a proof that is similar in nature to those for the other peak statistics.

Theorem 6.1. *The statistic* pk *is shuffle compatible.*

Proof. By Corollary 3.2 part (2) it suffices to prove that if $\pi, \pi' \in L([m])$ and $\sigma \in L([n]+m)$ with $\operatorname{pk}(\pi) = \operatorname{pk}(\pi')$ then $\operatorname{pk}(\pi \sqcup\!\sqcup \sigma) = \operatorname{pk}(\pi' \sqcup\!\sqcup \sigma)$. For the permutations with $\operatorname{pk} \pi = p$ we will use the canonical set

$$\Pi = \{\pi \in L([m]) \mid \operatorname{Pk}(\pi) = \{2, 4, \dots, 2p\}\}$$

which contains exactly the permutations with p peaks which are as far to the left as possible. So if $\pi, \pi' \in \Pi$ then $\operatorname{Pk}(\pi) = \operatorname{Pk}(\pi')$. It follows that $\operatorname{Pk}(\pi \sqcup\!\sqcup \sigma) = \operatorname{Pk}(\pi' \sqcup\!\sqcup \sigma)$ since Pk is shuffle compatible. Therefore $\operatorname{pk}(\pi \sqcup\!\sqcup \sigma) = \operatorname{pk}(\pi' \sqcup\!\sqcup \sigma)$ and the conclusion of (ii) of the general approach holds.

Our measure of how close a permutation is to being in Π is $d : L([m]) \to \mathbb{N}$ given by

$$d(\pi) = \sum_{k \in \operatorname{Pk}(\pi)} k.$$

Note that among all permutations in $L([m])$ with $\operatorname{pk} \pi = p$, the ones in Π have the minimum possible d, namely $p(p+1)$. Our strategy will be to find a bijection between shuffles $\tau \in \pi \sqcup\!\sqcup \sigma$ and shuffles with an element of Π which preserves pk and lowers d by the proper amount, namely $d(\pi) - p(p+1)$.

To reduce a permutation to one in Π we will move its peaks to the left one position at a time. More specifically, if $\pi \notin \Pi$ then there is at least one position $j \geq 3$ such that $j \in \mathrm{Pk}(\pi)$, but $j-2 \notin \mathrm{Pk}(\pi)$. Thus there exists $\pi'' \in L([m])$ such that

$$\mathrm{Pk}(\pi'') = \Big(\mathrm{Pk}(\pi) \setminus \{j\}\Big) \cup \{j-1\}. \tag{6}$$

Since $d(\pi'') < d(\pi)$, it suffices to give a pk-preserving bijection between $\pi \sqcup\!\sqcup \sigma$ and $\pi'' \sqcup\!\sqcup \sigma$.

For each $\tau \in \pi \sqcup\!\sqcup \sigma$, factor $\tau = \tau^a\tau^b\tau^c$ where τ^b is the factor of τ between π_{j-2} and π_{j+1}, not including π_{j-2} and π_{j+1}. Then τ^a is the remaining initial factor of τ and τ^c is the final factor. Factor τ^b even further as

$$\tau^b = \sigma^a \pi_{j-1} \sigma^b \pi_j \sigma^c$$

so that $\sigma^a,\ \sigma^b,\ \sigma^c$ are the factors of σ that are between the corresponding elements of π. Note that it is possible for any or all of $\sigma^a,\ \sigma^b,\ \sigma^c$ to be empty.

Define a map $\Theta : \pi \sqcup\!\sqcup \sigma \to \pi'' \sqcup\!\sqcup \sigma$ by

$$\Theta(\tau) = \begin{cases} (\tau^a)''\pi''_{j-1}\pi''_j\sigma^a(\tau^c)'' & \text{if } \sigma^a \neq \emptyset \text{ and } \sigma^b = \sigma^c = \emptyset, \\ (\tau^a)''\sigma^c\pi''_{j-1}\pi''_j(\tau^c)'' & \text{if } \sigma^a = \sigma^b = \emptyset \text{ and } \sigma^c \neq \emptyset, \\ \Phi(\tau) & \text{otherwise} \end{cases} \tag{7}$$

where $(\tau^a)''$ is the unique permutation such that $\omega(\tau^a) = \omega((\tau^a)'')$ and $(\tau^c)''$ is the unique permutation such that $\omega(\tau^c) = \omega((\tau^c)'')$. It is clear from its definition that Θ is a bijection. So it only remains to show that Θ is pk preserving. Let s, t be such that $\tau_s = \pi_{j-2}$ and $\tau_t = \pi_{j+1}$ and set $\ell = |\sigma^a|$. Note that, as in the proof of Theorem 4.2, for $i \in [m+n] \setminus [s,t]$ we have $i \in \mathrm{Pk}(\tau)$ if and only if $i \in \mathrm{Pk}(\tau'')$. So to show that Θ is pk preserving we just need to concentrate on those peaks in $[s,t]$.

If $\sigma^a \neq \emptyset$ and $\sigma^b = \sigma^c = \emptyset$ then it is straightforward to check, using the cases from the proof of Theorem 4.2 for Pk that the only peaks of τ in the set $[s,t]$ occur as one of the following.

(a) Every peak of σ^a is a peak of τ.
(b) $s+1 \in \mathrm{Pk}(\tau)$ if and only if $1 \in \mathrm{Des}(\sigma^a)$ or $\ell = 1$.
(c) For $\ell \geq 2$: $s+\ell \in \mathrm{Pk}(\tau)$ if and only if $\ell - 1 \in \mathrm{Asc}(\sigma^a)$.
(d) $t-1$ is always a peak of τ.

We now compare this to the similar list for τ''.

(a) Every peak of σ^a is a peak τ''.
(b) $s+3 \in \mathrm{Pk}(\tau'')$ if and only if $1 \in \mathrm{Des}(\sigma^a)$ or $\ell = 1$.

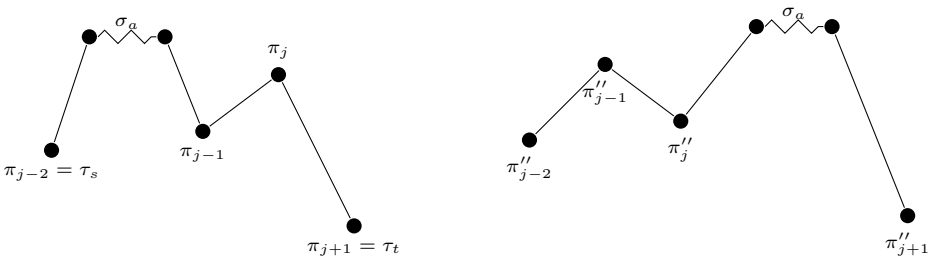

Fig. 3. An illustration of the case when $\sigma_a \neq \emptyset$ and $\sigma_b = \sigma_c = \emptyset$.

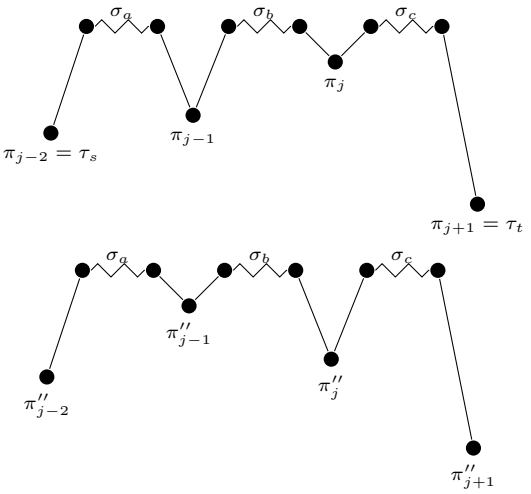

Fig. 4. An illustration of the case when σ_a, σ_b, and σ_c are all nonempty.

(c) For $\ell \geq 2$: $t-1 \in \mathrm{Pk}(\tau'')$ if and only if $\ell - 1 \in \mathrm{Asc}(\sigma^a)$.
(d) $s+1$ is always a peak of τ''.

Clearly these lists contain the same number of peaks. An illustration of this case is given in Fig. 3. In the figure, jagged lines represent a part of σ.

Next note that if $\sigma^a = \sigma^b = \emptyset$ and $\sigma^c \neq \emptyset$, then these two lists are swapped. So Θ is pk preserving in this case as well.

Now if τ^b is not in one of the previous two cases and σ^a, σ^b, and σ^c are not all simultaneously empty then one can check lists similar to those above to see that the peaks of both τ and τ'' in the range $[s,t]$ are exactly the peaks of σ^a, σ^b, and σ^c together with possibly their endpoints. An illustration of this case is given in Fig. 4. Set $\ell_a = |\sigma^a|$, $\ell_b = |\sigma^b|$, and $\ell_c = |\sigma^c|$ and suppose all three cardinalities are nonzero.

(a) Every peak of σ^a, σ^b and σ^c is a peak of τ.
(b) $s+1 \in \mathrm{Pk}(\tau)$ if and only if $1 \in \mathrm{Des}(\sigma^a)$ or $\ell_a = 1$.
(c) For $\ell_a \geq 2$: $s + \ell_a \in \mathrm{Pk}(\tau)$ if and only if $\ell_a - 1 \in \mathrm{Asc}(\sigma^a)$.

(d) $s + \ell_a + 2 \in \mathrm{Pk}(\tau)$ if and only if $1 \in \mathrm{Des}(\sigma^b)$ or $\ell_b = 1$.
(e) For $\ell_b \geq 2$: $s + \ell_a + \ell_b + 1 \in \mathrm{Pk}(\tau)$ if and only if $\ell_b - 1 \in \mathrm{Asc}(\sigma^b)$.
(d) $s + \ell_a + \ell_b + 3 \in \mathrm{Pk}(\tau)$ if and only if $1 \in \mathrm{Des}(\sigma^c)$ or $\ell_c = 1$.
(e) For $\ell_c \geq 2$: $s + \ell_a + \ell_b + \ell_c + 2 \in \mathrm{Pk}(\tau)$ if and only if $\ell_c - 1 \in \mathrm{Asc}(\sigma^c)$.

The list for τ'' is identical.

Finally, if $\sigma^a = \sigma^b = \sigma^c = \emptyset$, then $\mathrm{Pk}(\tau) \cap [s,t] = \{t-1\}$ and $\mathrm{Pk}(\tau'') \cap [s,t] = \{t-2\}$. Hence the number of peaks is again preserved. □

Theorem 6.2. *The statistics* lpk*,* rpk*,* epk*,* udr*, and* (udr, pk) *are shuffle compatible.*

The proofs for these statistics are based on the same idea as that of Theorem 6.1, but additional variants of the bijection used there are needed. We again use Corollary 3.2 part (2). So it suffices to show that if $\pi, \pi' \in L([m])$ and $\sigma \in L([n]+m)$ with $\mathrm{St}(\pi) = \mathrm{St}(\pi')$ then $\mathrm{St}(\pi \sqcup\!\sqcup \sigma) = \mathrm{St}(\pi' \sqcup\!\sqcup \sigma')$ for each $\mathrm{St} \in \{\mathrm{lpk}, \mathrm{rpk}, \mathrm{epk}, \mathrm{udr}, (\mathrm{udr}, \mathrm{pk})\}$. The main tools for this proof are the bijection Θ of Theorem 6.1 and other similar bijections that preserve these statistics and allow us to replace π with a permutation that is closer to being in our chosen set of canonical permutations, as outlined in the general approach.

Proof for lpk**.** To obtain the shuffle compatibility of lpk we reduce permutations with $\mathrm{lpk}\,\pi = p$ to the canonical set

$$\Pi = \{\pi \in L([m]) \mid \mathrm{Lpk}(\pi) = \{1, 3, 5, \ldots, 2p-1\}\}.$$

The proof that two permutations in this set have the same lpk is similar to the analogous statement for pk and so is omitted. We use a measure d similar to that for pk, but summing over left peaks instead. So the minimal value for a permutation with $\mathrm{lpk}(\pi) = p$ is

$$d(\pi) = \sum_{k \in \mathrm{Lpk}(\pi)} k = p^2.$$

If $\pi \notin \Pi$ then there exists a position $j \geq 2$ such that $j \in \mathrm{Lpk}(\pi)$, but $j-2 \notin \mathrm{Lpk}(\pi)$. Let π'' be any permutation such that

$$\mathrm{Lpk}(\pi'') = (\mathrm{Lpk}(\pi) \setminus \{j\}) \cup \{j-1\}. \tag{8}$$

Then it suffices to give a bijection $\widetilde{\Theta} : \pi \sqcup\!\sqcup \sigma \to \pi'' \sqcup\!\sqcup \sigma$ that reduces d and is lpk preserving. If $j \geq 3$, then the bijection of the above proof for pk suffices. Thus, assume $j = 2$. To construct $\widetilde{\Theta}$ we proceed in a manner similar to the proof of Theorem 5.2.

Set $\widetilde{\pi} = 0\pi$ and $\widetilde{\pi}'' = 0\pi''$ and use the notation

$$S_\pi = \{\tau \in \pi \sqcup\!\sqcup \sigma \mid \tau_1 = \pi_1\}.$$

Then we have the following commutative diagram.

$$\begin{array}{ccccc} \pi \sqcup\!\sqcup \sigma & \xrightarrow{\quad\iota\quad} & S_{\widetilde{\pi}} & \subseteq & \widetilde{\pi} \sqcup\!\sqcup \sigma \\ \downarrow{\scriptstyle \widetilde{\Theta}} & & \downarrow{\scriptstyle \Theta} & & \\ \pi \sqcup\!\sqcup \sigma'' & \xrightarrow{\quad\iota''\quad} & S_{\widetilde{\pi}''} & \subseteq & \widetilde{\pi} \sqcup\!\sqcup \sigma'' \end{array}$$

Here ι is the bijection which identifies $\tau \in \pi \sqcup\!\sqcup \sigma$ with $0\tau \in S_{\widetilde{\pi}}$. The map ι'' is defined similarly. The map Θ is the pk-preserving bijection used in the proof of pk to move the peak at position 3 to position 2.

Then

$$\widetilde{\Theta} = \iota''^{-1} \circ \Theta \circ \iota \tag{9}$$

is the required bijection. It is clear that this map reduces d by 1 since Θ has this property. The injection $\Theta \circ \iota$ is lpk preserving because Θ is pk preserving and position 2 in $\iota(\tau)$ is a peak if and only if position 1 is a left peak of τ. And similarly, position 1 in $\iota''^{-1}\Theta(\iota(\tau)))$ is a left peak if and only if position 2 is a peak in $\Theta(\iota(\tau))$ which proves the claim and completes the demonstration for lpk. □

Proof for rpk**.** To obtain the shuffle compatibility of rpk we use an approach similar to that of lpk by changing our set of canonical permutations for π with $\mathrm{rpk}(\pi) = p$ to

$$\Pi = \{\pi \in L([m]) \mid \mathrm{Rpk}(\pi) = \{m, m-2, m-4, \ldots, m-2p\}\}.$$

We also change our measure d of how close a permutation is to being in Π to

$$d(\pi) = \sum_{k \in \mathrm{Rpk}(\pi)} (m-k).$$

We have that the minimal value for a permutation with $\mathrm{rpk}(\pi) = p$ is $d(\pi) = p(p+1)$ and if two permutations $\pi_1, \pi_2 \in \Pi$ satisfy $\mathrm{Rpk}(\pi_1) = \mathrm{Rpk}(\pi_2)$, then by Theorem 4.2 they satisfy the conclusion of (ii) of the general approach.

If $\pi \notin \Pi$ then there exists a position $j \leq m-1$ such that $j \in \mathrm{Rpk}(\pi)$, but $j+2 \notin \mathrm{Rpk}(\pi)$. Let π'' be any permutation such that

$$\mathrm{Rpk}(\pi'') = (\mathrm{Rpk}(\pi) \setminus \{j\}) \cup \{j+1\}.$$

The remainder of the proof follows the same lines as for lpk except we move peaks to the right instead of left using the inverse bijections. □

Proof for epk**.** The shuffle compatibility of epk follows from the combination of bijections in the proofs for pk, lpk, and rpk. We use the canonical set of permutations

$$\Pi = \{\pi \in L([m]) \mid \mathrm{Epk}(\pi) = \{1, 3, 5, \ldots, 2p-1\} \text{ for some } p \geq 0\}$$

and measure

$$d(\pi) = \sum_{k \in \mathrm{Epk}(\pi)} k.$$

We move all peaks or right peaks as far to the left as possible using the bijections from pk, lpk and inverse of the bijection from rpk can be used to move a final ascent to the left. The remainder of the proof is analogous to those of lpk and rpk. □

Proof for udr**.** As a first step, we observe that for a permutation $\pi \in L([m])$, $m \geq 1$, we have

$$\mathrm{udr}(\pi) = 2\,\mathrm{pk}(0\pi) + \chi^+(0\pi) \tag{10}$$

since every peak of 0π involves two distinct runs, and $\chi^+(0\pi)$ accounts for the possibilities of either a final increasing run or that $|\pi| = 1$. There is nothing to prove if $\pi = \emptyset$ and the proof for $|\pi| = 1$ is trivial, so we will assume for the remainder of this proof that $m \geq 2$. In this case equation (10) simplifies slightly to

$$\mathrm{udr}(\pi) = 2\,\mathrm{lpk}(\pi) + \chi^+(\pi) \tag{11}$$

since the left peaks of π are the peaks of 0π. Considering this equation modulo two we see that the value of $\mathrm{udr}(\pi)$ determines both $\mathrm{lpk}(\pi)$ and $\chi^+(\pi)$, as well as conversely.

Just as for lpk, our canonical set for permutations with a $\mathrm{udr}\,\pi = 2p + \chi^+(\pi)$ is

$$\Pi = \{\pi \in L([m]) \mid \mathrm{Lpk}(\pi) = \{1, 3, 5, \ldots 2p-1\}\}.$$

Take two permutations $\pi, \pi' \in \Pi$ with the same udr. So, as discussed in the previous paragraph, $\chi^+(\pi) = \chi^+(\pi')$. It follows that $\mathrm{Lpk}(\pi) = \mathrm{Lpk}(\pi')$. So for any $\sigma \in L([n]+m)$ we have

$$\mathrm{udr}(\pi \sqcup\!\sqcup \sigma) = \mathrm{udr}(\pi' \sqcup\!\sqcup \sigma)$$

since the bijection Φ preserves both Lpk and χ^+. Therefore part (ii) of the general approach is satisfied.

Our measure of how close a permutation is to being in the canonical set will be the same as it was for lpk,

$$d(\pi) = \sum_{k \in \mathrm{Lpk}(\sigma)} k.$$

A permutation with $\mathrm{lpk}(\pi) = p$ is in this canonical set if and only if $d(\pi) = p^2$, which is the minimal value for a permutation with $\mathrm{udr}(\pi) = 2p + \chi^+(\pi)$. Since $\mathrm{udr}(\pi)$ is determined by $\mathrm{lpk}(\pi)$ and $\chi^+(\pi)$, to complete the proof it will suffice to show that the bijection $\widetilde{\Theta}$ from (9) used in the demonstration for lpk also preserves the statistic χ^+. Note first that by the definition of $\widetilde{\Theta}$ that it suffices to check that Θ itself is χ^+ preserving. This is because ι and ι'' essentially prepend a 0 to a permutation and then remove it. This does not affect the order relation between the final two elements of a shuffle since we have assumed that $|\pi| \geq 2$.

Since definition (7) for Θ uses the bijection Φ, we first show that Φ is χ^+ preserving when applied to $\pi, \pi'' \in L([m])$ that satisfy $\mathrm{udr}(\pi) = \mathrm{udr}(\pi'')$. As previously noted, the assumption about udr implies $\chi^+(\pi) = \chi^+(\pi'')$. Thus, the final two positions of π and π'' must satisfy the same order relation. It follows that for a shuffle $\tau \in \pi \sqcup\!\sqcup \sigma$, that replacing π with π'' to obtain $\tau' = \Phi(\tau)$ does not change the order relation of the final two positions of τ. This means that χ^+ is also preserved under Φ.

Now assume that $\pi \notin \Pi$. Choose π'' as in equation (8) with the additional restriction that $\mathrm{udr}(\pi) = \mathrm{udr}(\pi'')$. This implies $\chi^+(\pi'') = \chi^+(\pi)$. Let $\tau \in \pi \sqcup\!\sqcup \sigma$. If $j \leq m-2$ or $\tau_{m+n} \in \sigma$, it is clear from the definition of Θ given in (7) that $\chi^+(\tau) = \chi^+(\widetilde{\Theta}(\tau))$ and χ^+ is preserved.

It therefore remains to check that Θ is χ^+ preserving in the case that $j = m-1$ and $\tau_{m+n} = \pi_m$. Since we have already checked that Φ preserves χ^+, we only need to deal with the first two cases in the definition of Θ.

- If $\sigma^a \neq \emptyset$, $\sigma^b = \sigma^c = \emptyset$, then we have

$$\sigma^a \pi_{m-2}\pi_{m-1}\pi_m \overset{\Theta}{\mapsto} \pi''_{m-2}\pi''_{m-1}\sigma^a\pi''_m$$

  so that both shuffles have a descent at position $m+n-1$.
- If $\sigma^a = \sigma^b = \emptyset$, $\sigma^c \neq \emptyset$, then we have

$$\pi_{m-2}\pi_{m-1}\sigma^c\pi_m \overset{\Theta}{\mapsto} \sigma^c\pi''_{m-2}\pi''_{m-1}\pi''_m.$$

  Since π_{m-1} is a peak we have $\chi^+(\pi'') = \chi^+(\pi) = 0$. So, again, both shuffles have a descent at position $m+n-1$

From this we can conclude that the bijections Θ, and hence $\widetilde{\Theta}$, used in the proofs for the statistics pk and lpk respectively are also udr preserving. □

Proof for $(\mathrm{udr}, \mathrm{pk})$**.** For any permutation π with $|\pi| \geq 2$, we can write

$$\mathrm{lpk}(\pi) = \mathrm{pk}(\pi) + \chi^-(\pi)$$

So (11) becomes

$$\mathrm{udr}(\pi) = 2\,\mathrm{pk}(\pi) + 2\chi^-(\pi) + \chi^+(\pi). \tag{12}$$

By a parity argument like the one used for (11) we see that the value of $(\mathrm{udr}(\pi), \mathrm{pk}(\pi))$ uniquely determines both χ^-, and χ^+.

Let

$$\Pi_0 = \{\pi \in L([m]) \mid \mathrm{Lpk}(\pi) = \{2, 4, \ldots 2p\} \text{ for some } p \geq 0\}$$
$$\Pi_1 = \{\pi \in L([m]) \mid \mathrm{Lpk}(\pi) = \{1, 3, 5, \ldots 2p-1\} \text{ for some } p \geq 1\}$$

We then use the canonical set the disjoint union

$$\Pi = \Pi_0 \sqcup \Pi_1.$$

Note that if $\pi, \pi' \in \Pi$ satisfy $(\mathrm{udr}(\pi), \mathrm{pk}(\pi)) = (\mathrm{udr}(\pi'), \mathrm{pk}(\pi'))$ then, by the observation in the first paragraph of the proof, they are either both in Π_0 or both in Π_1. It follows that $\mathrm{Lpk}(\pi) = \mathrm{Lpk}(\pi')$. Now apply the bijection $\Phi : \pi \uplus \sigma \to \pi' \uplus \sigma$ where we have shown earlier that Φ preserves Lpk and χ^+. Also, the assumption on π and π' implies $\chi^-(\pi) = \chi^-(\pi')$. It is easy to prove that in this case Φ preserves χ^-. It follows that $(\mathrm{udr}, \mathrm{pk})$ is preserved by Φ and part (ii) of our method is satisfied.

Now set

$$d(\pi) = \sum_{k \in \mathrm{Lpk}(\pi)} k.$$

If $\pi \notin \Pi$ then we use the map Θ from the proof for pk to map $\pi \uplus \sigma$ to $\pi'' \uplus \sigma$ where π'' is given by (6), as long as j can be chosen with $j > 3$, or $j = 3$ and $\pi_1 < \pi_2$. If the only possible j value is $j = 3$ and $\pi_1 > \pi_2$ then we do not apply Θ, and we do not need to do so since $\pi \in \Pi_1$. It follows that we can always choose π'' so that $\chi^-(\pi) = \chi^-(\pi'')$. One can now show that in this case Θ preserves χ^- similarly to the proof that the map preserves χ^+. Since Θ also preserves pk, it preserves the pair $(\mathrm{udr}, \mathrm{pk})$ and we are done. □

7. Future work

There are still many open questions to be answered in this relatively new line of inquiry. The first natural question is whether a proof similar to those above can be given for the statistic $(\mathrm{udr}, \mathrm{pk}, \mathrm{des})$, which was conjectured to be shuffle compatible in [2].

Question 7.1. *Can a bijective proof for the statistic* $(\mathrm{udr}, \mathrm{pk}, \mathrm{des})$ *be given that follows the general approach given in this article?*

Such a proof will require a more careful analysis of the common aspects of the bijection used for des and for pk. The fundamental obstacle in our approach to this triple statistic

is that our bijection for pk requires moves that may not preserve the number of descents in the permutation.

Other notions of shuffle compatibility were introduced by Grinberg in [4], so one could ask whether the same general approach can be used to give bijective proofs for his shuffle compatibility analogues. One example is as follows.

Definition 7.2. A permutation statistic St is called *left shuffle compatible* if for any two disjoint nonempty permutations π and σ with the property that $\pi_1 > \sigma_1$, the distribution

$$\{\{\mathrm{St}(\tau) \mid \tau \in \pi \sqcup\!\sqcup \sigma,\ \tau_1 = \pi_1\ \}\}$$

depends only on $|\pi|$, $|\sigma|$, $\mathrm{St}(\pi)$, and $\mathrm{St}(\sigma)$. □

An analogous definition can be given for right shuffle compatibility. Note that an analogous set of shuffles appeared naturally in equation (4). Our theory here would need to be modified as even Lemma 3.1 no longer holds. The bijections used there can take a shuffle starting with $\tau_1 = \pi_1$ and swap it with one with $\tau_1 = \sigma_1$ which is no longer in the set of left shuffles.

The existence of a shuffle compatible permutation statistic that is not a descent statistic as constructed by Oğuz in [8] raises the question as to how one would approach giving bijective proofs for such statistics. The proof for the statistic in [8] is by an exhaustive computation for all permutations of length 4 or less. Note that for non-descent statistics our approach no longer works. Indeed, Lemma 3.1 only holds for descent statistics and so we lose the power of Corollary 3.2.

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